Trinity College

WA Exams Practice Paper B, 2015

Question/Answer Booklet

MATHEMATICS SPECIALIST UNITS 1 AND 2

Section Two: Calculator-assumed

SOLUTIONS

Student Number: In figures

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time for section: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
			Total	150	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

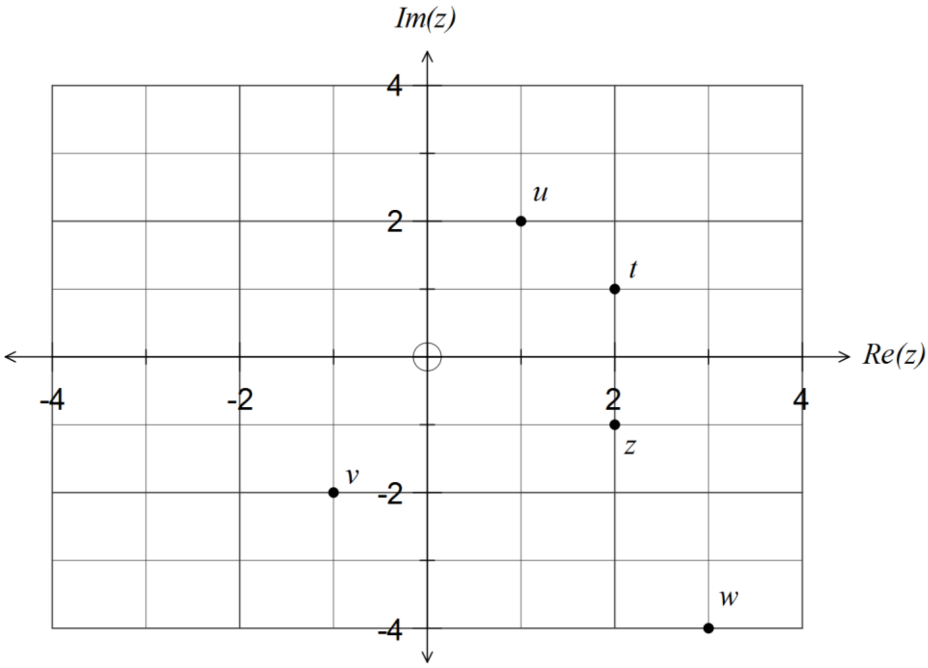
Section Two: Calculator-assumed **(98 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 8 **(4 marks)**

The Argand diagram below shows the complex number z. On the same diagram plot and label the four complex numbers given by $t=\overline{z}$, $u=i\,z$, $v=i^3\,z$ and $w=z^2$.



$t=\overline{z}$
$=2+i$

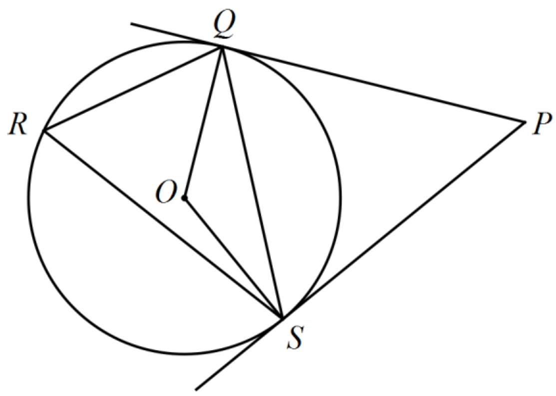
$u=i\,z$
$=i(2-i)=1+2i$

$v=i^3 z$
$=i^2 u=-u=-1-2i$

$w=z^2$
$=(2-i)(2-i)=4-4i+1=3-4i$

Question 9 **(8 marks)**

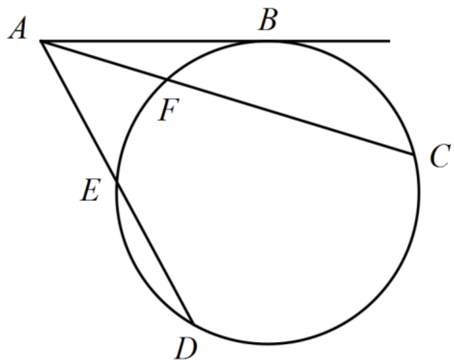
(a) In the diagram below, points Q, R and S lie on a circle with centre O, with tangents from P touching the circle at Q and S. If $\angle OSQ = 28°$, determine the size of angles QRS and QPS. (4 marks)



$\angle OQS = 28$
$\angle QOS = 180 - 28 - 28$
$= 124$
$\angle QRS = 124 \div 2$
$= 62°$

$\angle PSQ = 90 - 28$
$= 62$
$\angle PQS = \angle PSQ$
$= 62$
$\angle QPS = 180 - 62 - 62$
$= 56°$

(b) In the diagram below, points B, C, D, E and F lie on a circle, and AB is a tangent to the circle at point B. If $AB = 10$ cm, $AF = 6$ cm and $ED = 8$ cm, determine the exact lengths of FC and AE. (4 marks)



$AB^2 = AF.AC = AE.AD$

$x = FC$
$10^2 = 6(6 + x) \Rightarrow x = FC = \frac{32}{3}$

$y = AE$
$10^2 = y(y + 8) \Rightarrow y = AE = 2\sqrt{29} - 4$

Question 10 **(7 marks)**

Two vectors are given by $\mathbf{a} = -3\mathbf{i} + (k-2)\mathbf{j}$ and $\mathbf{b} = 4\mathbf{i} + (k+2)\mathbf{j}$.

(a) Determine the value(s) of k if $\mathbf{a}$ and $\mathbf{b}$ are

(i) parallel. (2 marks)

$$\frac{-3}{4} = \frac{k-2}{k+2}$$
$$-3k - 6 = 4k - 8$$
$$7k = 2$$
$$k = \tfrac{2}{7}$$

(ii) perpendicular. (3 marks)

$$\begin{bmatrix} -3 \\ k-2 \end{bmatrix} \bullet \begin{bmatrix} 4 \\ k+2 \end{bmatrix} = 0$$
$$-12 + (k-2)(k+2) = 0$$
$$k^2 = 16$$
$$k = \pm 4$$

(b) If $k = 3$, determine the angle between $\mathbf{a}$ and $\mathbf{b}$ to the nearest degree. (2 marks)

$\mathbf{a} = -3\mathbf{i} + \mathbf{j}$

$\mathbf{b} = 4\mathbf{i} + 5\mathbf{j}$

Using CAS, angle is 110°

Question 11 **(7 marks)**

A triangle has vertices $A\,(1, 1)$, $B\,(3, 1)$ and $C\,(3, 4)$.

(a) Triangle ABC is transformed to $A'\,(1, -1)$, $B'\,(3, -1)$ and $C'\,(3, -4)$. Describe this transformation geometrically and state the 2×2 matrix that will transform ABC to $A'B'C'$. (2 marks)

> Reflection in the line $y = 0$.
>
> $$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(b) Triangle $A'B'C'$ is dilated by a scale factor of ten about the origin and then rotated 30° clockwise about the origin to triangle $A''B''C''$.

(i) Determine the single 2×2 matrix that will transform $A'B'C'$ to $A''B''C''$. (3 marks)

> $$\begin{bmatrix} \cos(-30) & -\sin(-30) \\ \sin(-30) & \cos(-30) \end{bmatrix}\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$
> $$= \begin{bmatrix} 5\sqrt{3} & 5 \\ -5 & 5\sqrt{3} \end{bmatrix}$$

(ii) Determine the area of triangle $A''B''C''$. (2 marks)

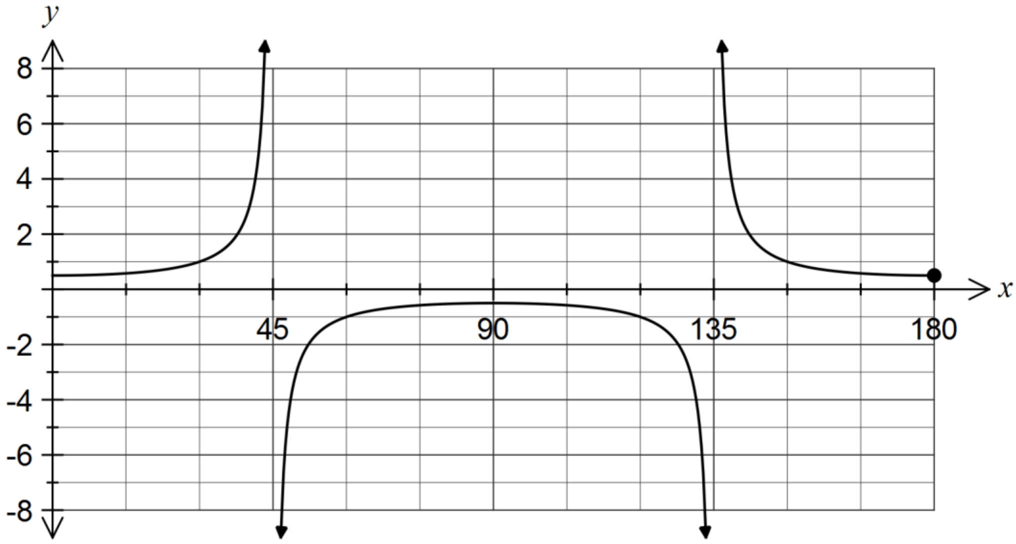
> Original area of ABC is 3 sq units.
>
> Reflection and rotation will not change area, but dilation will increase area by a factor of 10^2.
>
> Area is 300 sq units.

Question 12 **(7 marks)**

(a) Prove that $\dfrac{\tan^2\theta}{1+\sec\theta}+1=\sec\theta$. (3 marks)

$$\begin{aligned}\frac{\tan^2\theta}{1+\sec\theta}+1&=\frac{\sec^2\theta-1}{1+\sec\theta}+\frac{1+\sec\theta}{1+\sec\theta}\\&=\frac{(\sec\theta+1)(\sec\theta-1)+(\sec\theta+1)}{\sec\theta+1}\\&=\frac{(\sec\theta+1)(\sec\theta-1+1)}{\sec\theta+1}\\&=\frac{(\sec\theta+1)\sec\theta}{\sec\theta+1}\\&=\sec\theta\end{aligned}$$

(b) Draw the graph of $y=\dfrac{\sec(2\theta)}{2}$, $0\le\theta\le180°$. (2 marks)



(c) Solve $\dfrac{\tan^2 2\theta}{1+\sec 2\theta}+1=4$ for $0\le\theta\le180°$, giving your solutions to one decimal place.

(2 marks)

$$\frac{\tan^2 2\theta}{1+\sec 2\theta}+1=4$$

$$\sec 2\theta=4$$

$$\frac{\sec 2\theta}{2}=2$$

From graph, $\theta=37.8°$, $\theta=142.2°$

Question 13 **(9 marks)**

(a) If ${}^{n}C_{5} = {}^{n}C_{13}$, determine the value of n. (1 mark)

$$5+13=18$$

(b) Fifteen points lie in a plane, such that no three of them are collinear. How many triangles could be drawn using these points as vertices? (2 marks)

$${}^{15}C_{3} = 455$$

(c) Determine the number of diagonals that could be drawn inside a regular polygon with fifteen sides. (2 marks)

$$\begin{aligned} {}^{15}C_{2} - 15 &= 105 - 15 \\ &= 90 \end{aligned}$$

(Subtract 15 as adjacent pairs of vertices selected are not diagonals)

(d) A student has six coins with denominations 5 cents, 10 cents, 20 cents 50 cents, one dollar and two dollars. Determine how many different sums of money, greater than five cents, which the student can make with these coins. (4 marks)

With one coin: ${}^{6}C_{1} - 1 = 5$

With two coins: ${}^{6}C_{2} = 15$

Etc, etc.

Require $\sum_{r=1}^{6} {}^{6}C_{r} - 1 = 2^{6} - 1 - 1$

$$= 62$$

Question 14 **(8 marks)**

Relative to the origin, a small particle A is moving with velocity $-4\mathbf{i}+7\mathbf{j}$ ms^{-1} and another small particle B is moving with velocity $3\mathbf{i}+\mathbf{j}$ ms^{-1}.

(a) Calculate the angle between the velocities of A and B, rounding your answer to two decimal places. (2 marks)

$$\cos^{-1}\frac{(-4)(3)+(7)(1)}{\sqrt{65}\sqrt{10}} \approx 101.31\ (2\text{dp}) \quad \textit{or using CAS}$$

(b) Determine the velocity of B relative to A. (2 marks)

$$_B\mathbf{v}_A = \begin{bmatrix}3\\1\end{bmatrix} - \begin{bmatrix}-4\\7\end{bmatrix} = \begin{bmatrix}7\\-6\end{bmatrix}$$

(c) What is the exact speed of B relative to A? (1 mark)

$$|\,_B\mathbf{v}_A| = \sqrt{7^2+(-6)^2} = \sqrt{85}$$

(d) The velocity of a third small particle C relative to B is $5\mathbf{i}-4\mathbf{j}$. What is the exact speed of C relative to the origin? (3 marks)

$$_C\mathbf{v}_B = \begin{bmatrix}x\\y\end{bmatrix} - \begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}5\\-4\end{bmatrix} \Rightarrow \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}8\\-3\end{bmatrix}$$

$$\left\|\begin{bmatrix}8\\-3\end{bmatrix}\right\| = \sqrt{8^2+(-3)^2} = \sqrt{73}$$

Question 15 **(8 marks)**

(a) A group of students who had gained a distinction in either an English competition or a Chemistry competition were called to a meeting. 17 of the students had a distinction in the English competition, 14 students had a distinction in the Chemistry competition and 11 had a distinction in both. Determine how many students were in the group. (2 marks)

$$17 + 14 - 11 = 20$$

(b) Determine the number of different ways it is possible to order the letters PQQRRSSS. (2 marks)

$$\frac{8!}{2!2!3!} = 1680$$

(c) A six character password is to be made by selecting, without repetition, from the three lowercase letters a, b and c, the four uppercase letters D, E, F and G, and the five symbols @, #, $, % and *.

Determine the number of passwords that can be made if

(i) there must be equal numbers of lowercase letters, uppercase letters and symbols. (2 marks)

$${}^3C_2 \times {}^4C_2 \times {}^5C_2 \times 6! = 129600$$

(ii) there must be equal numbers of lowercase letters and uppercase letters and at least one symbol. (2 marks)

One LCase & one UCase:
$${}^3C_1 \times {}^4C_1 \times {}^5C_4 \times 6! = 43200$$

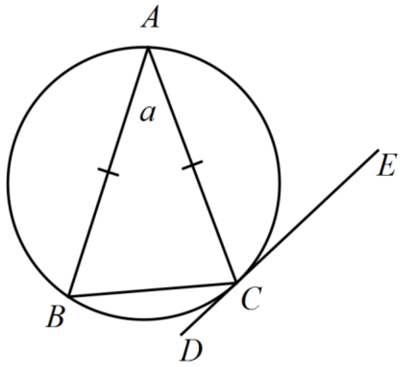
Two of each: ans(i)

Total: $129600 + 43200 = 172800$

Question 16 **(8 marks)**

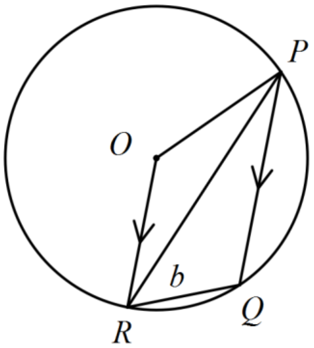
(a) Determine the size of angles a and b in the diagrams below.

(i) A, B and C lie on a circle. DE is a tangent at C, $AB = AC$ and $\angle DCA = 105°$. (2 marks)



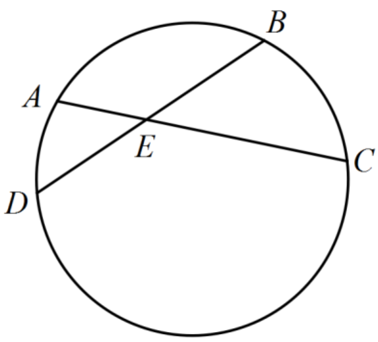
$$\angle ABC = \angle ECA$$
$$= 180 - 105$$
$$= 75$$
$$\angle ACB = \angle ABC$$
$$= 75$$
$$a = 180 - 75 - 75$$
$$a = 30°$$

(ii) P, Q and R lie on circle with centre O. PQ is parallel to OR and $\angle ORQ = 66°$. (3 marks)



$$\angle OQR = 66$$
$$\angle ROQ = 180 - 66 - 66$$
$$= 48$$
$$\angle RPQ = 48 \div 2$$
$$= 24$$
$$\angle RQP = 180 - 66$$
$$= 114$$
$$b = 180 - 114 - 24$$
$$b = 42°$$

(b) In the circle shown (not to scale) chords AC and BD intersect at E. If $AE = x$, $BE = 2x$, $CE = 5x + 1$ and $DE = 3x - 1$, determine the length x. Justify your answer. (3 marks)



Using intersecting chord theorem

$$x(5x+1) = 2x(3x-1)$$
$$x = 0, \; x = 3$$

Hence $x = 3$.

Question 17 **(8 marks)**

(a) If $P = \begin{bmatrix} 2 & -1 \\ -2 & n \end{bmatrix}$, where $n \in \mathbb{N}$, and $P^2 - 2P = \begin{bmatrix} 2 & a \\ b & 5 \end{bmatrix}$, determine the values of n, a and b.

(4 marks)

$$P^2 - 2P = \begin{bmatrix} 2 & -1 \\ -2 & n \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -2 & n \end{bmatrix} - 2\begin{bmatrix} 2 & -1 \\ -2 & n \end{bmatrix}$$
$$= \begin{bmatrix} 2 & -n \\ -2n & n^2 - 2n + 2 \end{bmatrix}$$

$$n^2 - 2n + 2 = 5$$
$$n^2 - 2n - 3 = 0$$
$$(n+1)(n-3) = 0 \Rightarrow n = 3 \ (n \in \mathbb{N})$$

$$a = -3$$
$$b = -6$$

(b) Let matrix $M = \begin{bmatrix} 3 & -6 \\ -1 & 3 \end{bmatrix}$.

(i) Determine the inverse of matrix M. (1 mark)

$$M^{-1} = \frac{1}{9-6}\begin{bmatrix} 3 & 6 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ \frac{1}{3} & 1 \end{bmatrix}$$

(ii) Express the equations $3x - 6y - 27 = 0$ and $11 + 3y - x = 0$ in matrix form.

(1 mark)

$$\begin{bmatrix} 3 & -6 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 27 \\ -11 \end{bmatrix}$$

(iii) Show use of your answer from (i) to solve the matrix equation in (ii). (2 marks)

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ \frac{1}{3} & 1 \end{bmatrix} \times \begin{bmatrix} 27 \\ -11 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

Question 18 **(6 marks)**

Prove the following identities:

(a) $\dfrac{1+\sin\theta}{\cos\theta}=\dfrac{\cos\theta}{1-\sin\theta}$ (3 marks)

$$\begin{aligned}
LHS &= \frac{1+\sin\theta}{\cos\theta} \\
&= \frac{1+\sin\theta}{\cos\theta}\,\frac{1-\sin\theta}{1-\sin\theta} \\
&= \frac{1-\sin^2\theta}{\cos\theta(1-\sin\theta)} \\
&= \frac{\cos^2\theta}{\cos\theta(1-\sin\theta)} \\
&= \frac{\cos\theta}{1-\sin\theta}
\end{aligned}$$

(b) $\dfrac{\sin\theta+\tan\theta}{1+\cos\theta}=\tan\theta$ (3 marks)

$$\begin{aligned}
LHS &= \frac{\sin\theta+\tan\theta}{1+\cos\theta} \\
&= \frac{\dfrac{\cos\theta\sin\theta}{\cos\theta}+\dfrac{\sin\theta}{\cos\theta}}{1+\cos\theta} \\
&= \frac{\cos\theta\sin\theta+\sin\theta}{(1+\cos\theta)\cos\theta} \\
&= \frac{(1+\cos\theta)\sin\theta}{(1+\cos\theta)\cos\theta} \\
&= \tan\theta
\end{aligned}$$

Question 19 **(9 marks)**

(a) Consider the statement if $n^2 - 6n + 5$ is even, then n is odd.

(i) Write the contrapositive of this statement. (1 mark)

If n is not odd, then $n^2 - 6n + 5$ is not even.

(ii) Hence, or otherwise, prove that the statement is true. (3 marks)

n not odd $\Rightarrow$ n is even and of the form $2m$, where m is an integer.

$(2m)^2 - 6(2m) + 5 = 4m^2 - 12m + 5$

$= 2(2m^2 - 6m + 2) + 1$ (hence always odd)

We have proven that the contrapositive is true, and thus the original statement is also true.

(b) The variables k and m are both positive integers such that $m^2 + 3 = 2k$.

(i) Explain why m must always be odd. (1 mark)

$2k$ is even and hence $m^2 + 3$ must be even.
Since 3 is odd then m^2 must be odd, and so m must also be odd.

(ii) Using the fact that an odd integer can be written in the form $2n + 1$, or otherwise, prove that k is always the sum of three square numbers. (4 marks)

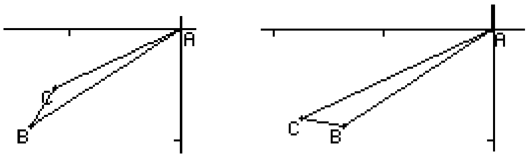
$$\begin{aligned}2k &= m^2 + 3 \\ &= (2n+1)^2 + 3 \\ &= 4n^2 + 4n + 4 \\ k &= 2n^2 + 2n + 2 \\ &= n^2 + n^2 + 2n + 1 + 1 \\ &= n^2 + (n+1)^2 + 1^2\end{aligned}$$

Question 20 **(9 marks)**

A current of constant velocity 2 km/h runs parallel to the sides of a straight section of a shipping canal. The skipper of a small boat motors at a speed of 8 km/h on a bearing of 237º in the canal. Due to the current, the boat appears to be moving directly towards a pontoon on the canal edge, which is on a bearing of 245º from the boat.

If the speed of the boat over the canal floor is less than its actual speed through the water determine

(a) the bearing on which the current is flowing. (5 marks)



AB is boat relative to water, BC is water relative to earth, AC is boat relative to earth.

2 solutions to triangle with AB=8, BC=2 and BAC=245-237=8º.

Must choose LH diagram (C obtuse) as given that |AC|<|AB|.

$$\frac{\sin(C)}{8} = \frac{\sin(8)}{2}$$
$$C = 146.2°$$
$$B = 180 - 146.2 - 8 = 25.8°$$

Bearing is $57 - 25.8 = 031°$ to nearest degree.

(b) the time the boat will take to reach the pontoon, to the nearest minute, if the pontoon is 650m away. (4 marks)

$$v^2 = \text{AC}^2 = 2^2 + 8^2 - 2 \times 2 \times 8 \times \cos(25.8)$$
$$v = 6.261 \text{ km/h}$$
$$t = 0.650 \div 6.261$$
$$t = 0.1038 \text{ hours}$$
$$t = 6.2 \approx 6 \text{ minutes}$$

End of questions

Additional working space

Question number: _________

Additional working space

Question number: _________

Additional working space

Question number: _________

Additional working space

Question number: _________